The Interaction Abstract Machine

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1 Linear Machines

Linear IAM[1]

Terms	$t,u ::= x \mid \lambda x.t \mid tu$	Contexts	$C,D ::= \langle\cdot\rangle \mid \lambda x.C \mid Ct \mid tC$
Tapes	$T ::= \epsilon \mid \bullet\cdot T \mid \circ\cdot T$	Directions	$d ::= \downarrow \mid \uparrow$
States	$q ::= (t,C,T,d)$	Initial States	$q_t ::= (t,\langle\cdot\rangle,\epsilon,\downarrow)$

Sub-term	Context	Tape		Sub-term	Context	Tape
$\underline{tu}$	C	T	$\to_{\bullet 1}$	$\underline{t}$	$C\langle\langle\cdot\rangle u\rangle$	$\bullet\cdot T$
$\underline{\lambda x.t}$	C	$\bullet\cdot T$	$\to_{\bullet 2}$	$\underline{t}$	$C\langle\lambda x.\langle\cdot\rangle\rangle$	T
$\underline{x}$	$C\langle\lambda x.D\rangle$	T	$\to_{\mathsf{var}}$	$\lambda x.D\langle x\rangle$	$\underline{C}$	$\circ\cdot T$
$\underline{\lambda x.D\langle x\rangle}$	C	$\circ\cdot T$	$\to_{\mathsf{bt2}}$	x	$\underline{C\langle\lambda x.D\rangle}$	T
t	$\underline{C\langle\langle\cdot\rangle u\rangle}$	$\bullet\cdot T$	$\to_{\bullet 3}$	tu	$\underline{C}$	T
t	$\underline{C\langle\lambda x.\langle\cdot\rangle\rangle}$	T	$\to_{\bullet 4}$	$\lambda x.t$	$\underline{C}$	$\bullet\cdot T$
t	$\underline{C\langle\langle\cdot\rangle u\rangle}$	$\circ\cdot T$	$\to_{\mathsf{arg}}$	$\underline{u}$	$C\langle t\langle\cdot\rangle\rangle$	T
u	$\underline{C\langle t\langle\cdot\rangle\rangle}$	T	$\to_{\mathsf{bt1}}$	$\underline{t}$	$C\langle\langle\cdot\rangle u\rangle$	$\circ\cdot T$

Linear Type IAM[2]

Types $A,B ::= \star \mid A\to B$ Type Contexts $\mathbb{A} ::= \langle\cdot\rangle \mid B\to\mathbb{A} \mid \mathbb{A}\to B$

Linear (or Affine) Typing Rules

$$\frac{}{(\Gamma,)\, x:A\vdash x:A} \qquad \frac{\Gamma,x:B\vdash t:A}{\Gamma\vdash\lambda x.t:B\to A} \qquad \frac{}{(\Gamma)\vdash\lambda x.t:\star} \qquad \frac{\Gamma\vdash t:B\to A \quad \Delta\vdash u:B}{\Gamma,\Delta\vdash tu:A}$$

$$\frac{\vdash t:B\to A \quad \ldots\vdash\ldots}{\vdash tu:\mathbb{A}\langle\star_\uparrow\rangle} \to_{\bullet 1} \frac{\vdash t:B\to\mathbb{A}\langle\star_\uparrow\rangle \quad \ldots\vdash\ldots}{\vdash tu:A} \qquad \frac{\vdash t:A}{\vdash\lambda x.t:B\to\mathbb{A}\langle\star_\uparrow\rangle} \to_{\bullet 2} \frac{\vdash t:\mathbb{A}\langle\star_\uparrow\rangle}{\vdash\lambda x.t:B\to A}$$

$$\frac{\vdash t:B\to\mathbb{A}\langle\star_\downarrow\rangle \quad \ldots\vdash\ldots}{\vdash tu:A} \to_{\bullet 3} \frac{\vdash t:B\to A \quad \ldots\vdash\ldots}{\vdash tu:\mathbb{A}\langle\star_\downarrow\rangle} \qquad \frac{\vdash t:\mathbb{A}\langle\star_\downarrow\rangle}{\vdash\lambda x.t:B\to A} \to_{\bullet 4} \frac{\vdash t:A}{\vdash\lambda x.t:B\to\mathbb{A}\langle\star_\downarrow\rangle}$$

$$\frac{\begin{array}{c}\overline{\vdash x:\mathbb{A}\langle\star_\uparrow\rangle}\\ \vdots\end{array}}{\vdash\lambda x.C\langle x\rangle:A\to B} \to_{\mathsf{var}} \frac{\begin{array}{c}\overline{\vdash x:A}\\ \vdots\end{array}}{\vdash\lambda x.C\langle x\rangle:\mathbb{A}\langle\star_\downarrow\rangle\to B} \qquad \frac{\begin{array}{c}\overline{\vdash x:A}\\ \vdots\end{array}}{\vdash\lambda x.C\langle x\rangle:\mathbb{A}\langle\star_\uparrow\rangle\to B} \to_{\mathsf{bt2}} \frac{\begin{array}{c}\overline{\vdash x:\mathbb{A}\langle\star_\downarrow\rangle}\\ \vdots\end{array}}{\vdash\lambda x.C\langle x\rangle:A\to B}$$

$$\frac{\vdash t:\mathbb{A}\langle\star_\downarrow\rangle\to B \quad \vdash u:A}{\vdash tu:B} \to_{\mathsf{arg}} \frac{\vdash t:A\to B \quad \vdash u:\mathbb{A}\langle\star_\uparrow\rangle}{\vdash tu:B} \qquad \frac{\vdash t:A\to B \quad \vdash u:\mathbb{A}\langle\star_\downarrow\rangle}{\vdash tu:B} \to_{\mathsf{bt1}} \frac{\vdash t:\mathbb{A}\langle\star_\uparrow\rangle\to B \quad \vdash u:A}{\vdash tu:B}$$

[1] The machine actually works also on affine λ-terms. Moreover, since the IAM implements *weak* head reduction, there are no linearity constraints for the subterm $\lambda x.t$ that will become the normal form (one can notice this fact from the type system).

[2] Type derivations are upside-down wrt to the term structure, then direction $\downarrow$ of the IAM becomes here $\uparrow$, and $\uparrow$ is $\downarrow$. The type environment Γ in the two axioms turns the linear system into an *affine* one. Type environments are omitted in the transition rules as they are not relevant to the description of the machine. It always holds that $A=\mathbb{A}\langle\star\rangle$.

2 Examples

Linear IAM.

	Sub-term	Context	Tape	Dir
	$\underline{(\lambda y.\lambda x.xy)\mathsf{I}(\lambda z.z)}$	$\langle\cdot\rangle$	ϵ	$\downarrow$
$\to_{\bullet 1}$	$\underline{(\lambda y.\lambda x.xy)\mathsf{I}}$	$\langle\cdot\rangle(\lambda z.z)$	$\bullet$	$\downarrow$
$\to_{\bullet 1}$	$\underline{\lambda y.\lambda x.xy}$	$\langle\cdot\rangle\mathsf{I}(\lambda z.z)$	$\bullet\cdot\bullet$	$\downarrow$
$\to_{\bullet 2}$	$\underline{\lambda x.xy}$	$(\lambda y.\langle\cdot\rangle)\mathsf{I}(\lambda z.z)$	$\bullet$	$\downarrow$
$\to_{\bullet 2}$	$\underline{xy}$	$(\lambda y.\lambda x.\langle\cdot\rangle)\mathsf{I}(\lambda z.z)$	ϵ	$\downarrow$
$\to_{\bullet 1}$	$\underline{x}$	$(\lambda y.\lambda x.\langle\cdot\rangle y)\mathsf{I}(\lambda z.z)$	$\bullet$	$\downarrow$
$\to_{\mathsf{var}}$	$\lambda x.xy$	$\underline{(\lambda y.\langle\cdot\rangle)\mathsf{I}(\lambda z.z)}$	$\circ\cdot\bullet$	$\uparrow$
$\to_{\bullet 4}$	$\lambda y.\lambda x.xy$	$\underline{\langle\cdot\rangle\mathsf{I}(\lambda z.z)}$	$\bullet\cdot\circ\cdot\bullet$	$\uparrow$
$\to_{\bullet 3}$	$(\lambda y.\lambda x.xy)\mathsf{I}$	$\underline{\langle\cdot\rangle(\lambda z.z)}$	$\circ\cdot\bullet$	$\uparrow$
$\to_{\mathsf{arg}}$	$\underline{\lambda z.z}$	$(\lambda y.\lambda x.xy)\mathsf{I}\langle\cdot\rangle$	$\bullet$	$\downarrow$
$\to_{\bullet 2}$	$\underline{z}$	$(\lambda y.\lambda x.xy)\mathsf{I}(\lambda z.\langle\cdot\rangle)$	ϵ	$\downarrow$
$\to_{\mathsf{var}}$	$\lambda z.z$	$\underline{(\lambda y.\lambda x.xy)\mathsf{I}\langle\cdot\rangle}$	$\circ$	$\uparrow$
$\to_{\mathsf{bt1}}$	$\underline{(\lambda y.\lambda x.xy)\mathsf{I}}$	$\langle\cdot\rangle(\lambda z.z)$	$\circ\cdot\circ$	$\downarrow$
$\to_{\bullet 1}$	$\underline{\lambda y.\lambda x.xy}$	$\langle\cdot\rangle\mathsf{I}(\lambda z.z)$	$\bullet\cdot\circ\cdot\circ$	$\downarrow$
$\to_{\bullet 2}$	$\underline{\lambda x.xy}$	$(\lambda y.\langle\cdot\rangle)\mathsf{I}(\lambda z.z)$	$\circ\cdot\circ$	$\downarrow$
$\to_{\mathsf{bt2}}$	x	$\underline{(\lambda y.\lambda x.\langle\cdot\rangle y)\mathsf{I}(\lambda z.z)}$	$\circ$	$\uparrow$
$\to_{\mathsf{arg}}$	$\underline{y}$	$(\lambda y.\lambda x.x\langle\cdot\rangle)\mathsf{I}(\lambda z.z)$	ϵ	$\downarrow$
$\to_{\mathsf{var}}$	$\lambda y.\lambda x.xy$	$\underline{\langle\cdot\rangle\mathsf{I}(\lambda z.z)}$	$\circ$	$\uparrow$
$\to_{\mathsf{arg}}$	$\underline{\mathsf{I}}$	$(\lambda y.\lambda x.xy)\langle\cdot\rangle(\lambda z.z)$	ϵ	$\downarrow$

Linear Type IAM.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\star\to\star\vdash x:\star_{\downarrow 16}\to\star_{\uparrow 6}}\quad\dfrac{}{y:\star\vdash y:\star_{\uparrow 17}}}{y:\star,x:\star\to\star\vdash xy:\star_{\uparrow 5}}}{y:[\star]\vdash\lambda x.xy:(\star_{\uparrow 15}\to\star_{\downarrow 7})\to\star_{\uparrow 4}}}{\vdash\lambda y.\lambda x.xy:\star_{\downarrow 18}\to(\star_{\uparrow 14}\to\star_{\downarrow 8})\to\star_{\uparrow 3}}\quad\dfrac{}{\vdash\mathsf{I}:\star_{\uparrow 19}}}{\vdash(\lambda y.\lambda x.xy)\mathsf{I}:(\star_{\uparrow 13}\to\star_{\downarrow 9})\to\star_{\uparrow 2}}\quad\dfrac{\dfrac{}{z:\star\vdash z:\star_{\uparrow 11}}}{\vdash\lambda z.z:\star_{\downarrow 12}\to\star_{\uparrow 10}}}{\vdash(\lambda y.\lambda x.xy)\mathsf{I}(\lambda z.z):\star_{\uparrow 1}}$$

3 Full Machines

IAM

LOGGED POSITIONS	$l ::= (t, C_n, L_n)$	TAPES	$T ::= \epsilon \mid \bullet\cdot T \mid l\cdot T$
LOGS	$L_0 ::= \epsilon \qquad L_{n+1} ::= l\cdot L_n$	DIRECTION	$d ::= \downarrow \mid \uparrow$
STATES	$q ::= (t, C, L, T, d)$	CONTEXTS	$C_0 ::= \langle\cdot\rangle \mid \lambda x.C_0 \mid C_0 t$
INITIAL STATES	$q_t ::= (t, \langle\cdot\rangle, \epsilon, \epsilon, \downarrow)$		$C_{n+1} ::= C_{n+1}t \mid \lambda x.C_{n+1} \mid tC_n$

Sub-term	Context	Log	Tape		Sub-term	Context	Log	Tape
$\underline{tu}$	C	L	T	$\to_{\bullet 1}$	$\underline{t}$	$C\langle\langle\cdot\rangle u\rangle$	L	$\bullet\cdot T$
$\underline{\lambda x.t}$	C	L	$\bullet\cdot T$	$\to_{\bullet 2}$	$\underline{t}$	$C\langle\lambda x.\langle\cdot\rangle\rangle$	L	T
$\underline{x}$	$C\langle\lambda x.D_n\rangle$	$L_n\cdot L$	T	$\to_{\mathsf{var}}$	$\lambda x.D_n\langle x\rangle$	$\underline{C}$	L	$(x,\lambda x.D_n,L_n)\cdot T$
$\underline{\lambda x.D_n\langle x\rangle}$	C	L	$(x,\lambda x.D_n,L_n)\cdot T$	$\to_{\mathsf{bt2}}$	x	$\underline{C\langle\lambda x.D_n\rangle}$	$L_n\cdot L$	T
t	$\underline{C\langle\langle\cdot\rangle u\rangle}$	L	$\bullet\cdot T$	$\to_{\bullet 3}$	tu	$\underline{C}$	L	T
t	$\underline{C\langle\lambda x.\langle\cdot\rangle\rangle}$	L	T	$\to_{\bullet 4}$	$\lambda x.t$	$\underline{C}$	L	$\bullet\cdot T$
t	$\underline{C\langle\langle\cdot\rangle u\rangle}$	L	$l\cdot T$	$\to_{\mathsf{arg}}$	$\underline{u}$	$C\langle t\langle\cdot\rangle\rangle$	$l\cdot L$	T
t	$\underline{C\langle u\langle\cdot\rangle\rangle}$	$l\cdot L$	T	$\to_{\mathsf{bt1}}$	$\underline{u}$	$C\langle\langle\cdot\rangle t\rangle$	L	$l\cdot T$

Sequence Type IAM

Linear Types	$A ::= \star \mid S \to A$	Linear Type Ctxs	$\mathbb{A} ::= \langle\cdot\rangle \mid S \to \mathbb{A} \mid \mathbb{S} \to A$
Sequence Types	$S ::= [A_1, \ldots, A_n]$	Sequence Type Ctxs	$\mathbb{S} ::= [A_1, \ldots, \mathbb{A}, \ldots, A_n]$

Sequence Typing Rules

$$\frac{}{x : [A] \vdash x : A} \qquad \frac{\Gamma, x : S \vdash t : A}{\Gamma \vdash \lambda x.t : S \to A} \qquad \frac{}{\vdash \lambda x.t : \star} \qquad \frac{\Gamma \vdash t : [A_1, .., A_n] \to A \quad [\Delta_i \vdash u : A_i]_{i \leq n}}{\Gamma \uplus_i \Delta_i \vdash tu : A}$$

$$\frac{\vdash t : S \to A \quad [\vdash]}{\vdash \color{red}{tu} : \mathbb{A}\langle\star_{\color{red}{\uparrow}}\rangle} \;\to_{\bullet 1}\; \frac{\vdash \color{red}{t} : S \to \mathbb{A}\langle\star_{\color{red}{\uparrow}}\rangle \quad [\vdash]}{\vdash tu : A} \qquad\qquad \frac{\vdash t : A}{\vdash \color{red}{\lambda x.t} : S \to \mathbb{A}\langle\star_{\color{red}{\uparrow}}\rangle} \;\to_{\bullet 2}\; \frac{\vdash \color{red}{t} : \mathbb{A}\langle\star_{\color{red}{\uparrow}}\rangle}{\vdash \lambda x.t : S \to A}$$

$$\frac{\vdash \color{blue}{t} : S \to \mathbb{A}\langle\star_{\color{blue}{\downarrow}}\rangle \quad [\vdash]}{\vdash tu : A} \;\to_{\bullet 3}\; \frac{\vdash t : S \to A \quad [\vdash]}{\vdash \color{blue}{tu} : \mathbb{A}\langle\star_{\color{blue}{\downarrow}}\rangle} \qquad\qquad \frac{\vdash \color{blue}{t} : \mathbb{A}\langle\star_{\color{blue}{\downarrow}}\rangle}{\vdash \lambda x.t : S \to A} \;\to_{\bullet 4}\; \frac{\vdash t : A}{\vdash \color{blue}{\lambda x.t} : S \to \mathbb{A}\langle\star_{\color{blue}{\downarrow}}\rangle}$$

$$\begin{array}{c} \overline{\vdash \color{red}{x} : \mathbb{A}\langle\star_{\color{red}{\uparrow}}\rangle_i}\; i \\ \vdots \\ \hline \vdash \lambda x.C\langle x\rangle : [\ldots, A_i, \ldots] \to A' \end{array} \;\to_{\mathsf{var}}\; \begin{array}{c} \overline{\vdash x : A_i}\; i \\ \vdots \\ \hline \vdash \color{blue}{\lambda x.C\langle x\rangle} : [\ldots, \mathbb{A}\langle\star_{\color{blue}{\downarrow}}\rangle_i, \ldots] \to A' \end{array}$$

$$\begin{array}{c} \overline{\vdash x : A_i}\; i \\ \vdots \\ \hline \vdash \color{red}{\lambda x.C\langle x\rangle} : [\ldots, \mathbb{A}\langle\star_{\color{red}{\uparrow}}\rangle_i, \ldots] \to A' \end{array} \;\to_{\mathsf{bt2}}\; \begin{array}{c} \overline{\vdash \color{blue}{x} : \mathbb{A}\langle\star_{\color{blue}{\downarrow}}\rangle_i}\; i \\ \vdots \\ \hline \vdash \lambda x.C\langle x\rangle : [\ldots, A_i, \ldots] \to A' \end{array}$$

$$\frac{\vdash \color{blue}{t} : [\ldots, \mathbb{A}\langle\star_{\color{blue}{\downarrow}}\rangle_i, \ldots] \to A' \quad \vdash_i u : A_i}{\vdash tu : A'} \;\to_{\mathsf{arg}}\; \frac{\vdash t : [\ldots, A_i, \ldots] \to A' \quad \vdash_i \color{red}{u} : \mathbb{A}\langle\star_{\color{red}{\uparrow}}\rangle_i}{\vdash tu : A'}$$

$$\frac{\vdash t : [\ldots, A_i, \ldots] \to A' \quad \vdash_i \color{blue}{u} : \mathbb{A}\langle\star_{\color{blue}{\downarrow}}\rangle_i}{\vdash tu : A'} \;\to_{\mathsf{bt1}}\; \frac{\vdash \color{red}{t} : [\ldots, \mathbb{A}\langle\star_{\color{red}{\uparrow}}\rangle_i, \ldots] \to A' \quad \vdash_i u : A_i}{\vdash tu : A'}$$

(Optimized) KAM[3]

Closures	Environments	Stacks	States
$c ::= (t, C, e)$	$e ::= \emptyset \mid [x{\leftarrow}c] \cup e$	$\pi ::= \epsilon \mid c \cdot \pi$	$q ::= (t, C, e, \pi)$

Initial States
$q_t ::= (t, \langle\cdot\rangle, \epsilon, \epsilon)$

Term	Ctx	Env	Stack		Term	Ctx	Env	Stack
$\color{red}{tu}$	C	e	π	$\to_{\mathsf{app}}$	$\color{red}{t}$	$C\langle\langle\cdot\rangle u\rangle$	e	$(u, C\langle t\langle\cdot\rangle\rangle, e) \cdot \pi$
$\color{red}{\lambda x.t}$	C	e	$c \cdot \pi$	$\to_{\mathsf{abs}}$	$\color{red}{t}$	$C\langle\lambda x.\langle\cdot\rangle\rangle$	$[x{\leftarrow}c] \cup e$	π
$\color{red}{x}$	C	$[x{\leftarrow}(t, D, e')] \cup e$	π	$\to_{\mathsf{var}}$	$\color{red}{t}$	D	e'	π
$\color{red}{tx}$	C	e	π	$\to_{\mathsf{appv}}$	$\color{red}{t}$	$C\langle\langle\cdot\rangle x\rangle$	e	$e(x) \cdot \pi$

[3]Whether the KAM is optimized or not depends on the inclusion of the rule $\to_{\mathsf{appv}}$.

4 Examples

IAM.

	Sub-term	Context	Log	Tape	Dir
	$(\lambda x.xx)(\lambda y.y)$	$\langle\cdot\rangle$	ϵ	ϵ	$\downarrow$
$\to_{\bullet 1}$	$\lambda x.xx$	$\langle\cdot\rangle(\lambda y.y)$	ϵ	$\bullet$	$\downarrow$
$\to_{\bullet 2}$	xx	$(\lambda x.\langle\cdot\rangle)(\lambda y.y)$	ϵ	ϵ	$\downarrow$
$\to_{\bullet 1}$	x	$(\lambda x.\langle\cdot\rangle x)(\lambda y.y)$	ϵ	$\bullet$	$\downarrow$
$\to_{\mathsf{var}}$	$\lambda x.xx$	$\langle\cdot\rangle(\lambda y.y)$	ϵ	$(x, \lambda x.\langle\cdot\rangle x, \epsilon) \cdot \bullet$	$\uparrow$
$\to_{\mathsf{arg}}$	$\lambda y.y$	$(\lambda x.xx)\langle\cdot\rangle$	$(x, \lambda x.\langle\cdot\rangle x, \epsilon)$	$\bullet$	$\downarrow$
$\to_{\bullet 2}$	y	$(\lambda x.xx)(\lambda y.\langle\cdot\rangle)$	$(x, \lambda x.\langle\cdot\rangle x, \epsilon)$	ϵ	$\downarrow$
$\to_{\mathsf{var}}$	$\lambda y.y$	$(\lambda x.xx)\langle\cdot\rangle$	$(x, \lambda x.\langle\cdot\rangle x, \epsilon)$	$(y, \lambda y.\langle\cdot\rangle, \epsilon)$	$\uparrow$
$\to_{\mathsf{bt1}}$	$\lambda x.xx$	$\langle\cdot\rangle(\lambda y.y)$	ϵ	$(x, \lambda x.\langle\cdot\rangle x, \epsilon) \cdot (y, \lambda y.\langle\cdot\rangle, \epsilon)$	$\downarrow$
$\to_{\mathsf{bt2}}$	x	$(\lambda x.\langle\cdot\rangle x)(\lambda y.y)$	ϵ	$(y, \lambda y.\langle\cdot\rangle, \epsilon)$	$\uparrow$
$\to_{\mathsf{arg}}$	x	$(\lambda x.x\langle\cdot\rangle)(\lambda y.y)$	$(y, \lambda y.\langle\cdot\rangle, \epsilon)$	ϵ	$\downarrow$
$\to_{\mathsf{var}}$	$\lambda x.xx$	$\langle\cdot\rangle(\lambda y.y)$	ϵ	$(x, \lambda x.x\langle\cdot\rangle, (y, \lambda y.\langle\cdot\rangle, \epsilon))$	$\uparrow$
$\to_{\mathsf{arg}}$	$\lambda y.y$	$(\lambda x.xx)\langle\cdot\rangle$	$(x, \lambda x.x\langle\cdot\rangle, (y, \lambda y.\langle\cdot\rangle, \epsilon))$	ϵ	$\downarrow$

SIAM.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : [[\star] \to \star] \vdash x : [\star_{\downarrow 10}] \to \star_{\uparrow 4}} \quad \dfrac{}{x : [\star] \vdash x : \star_{\uparrow 11}}}{x : [[\star] \to \star, \star] \vdash xx : \star_{\uparrow 3}}}{\vdash \lambda x.xx : [[\star_{\uparrow 9}] \to \star_{\downarrow 5}, \star_{\downarrow 12}] \to \star_{\uparrow 2}} \quad \dfrac{\dfrac{}{y : \star \vdash y : \star_{\uparrow 7}}}{\vdash \lambda y.y : [\star_{\downarrow 8}] \to \star_{\uparrow 6}} \quad \dfrac{}{\vdash \lambda y.y : \star_{\uparrow 13}}}{\vdash (\lambda x.xx)(\lambda y.y) : \star_{\uparrow 1}}$$

KAM.

	Sub-term	Context	Env.	Stack
	$(\lambda x.xx)(\lambda y.y)$	$\langle\cdot\rangle$	ϵ	ϵ
$\to_{\mathsf{app}}$	$\lambda x.xx$	$\langle\cdot\rangle(\lambda y.y)$	ϵ	$(\lambda y.y, (\lambda x.xx)\langle\cdot\rangle, \epsilon)$
$\to_{\mathsf{abs}}$	xx	$(\lambda x.\langle\cdot\rangle)(\lambda y.y)$	$[x{\leftarrow}(\lambda y.y, (\lambda x.xx)\langle\cdot\rangle, \epsilon)] =: e$	ϵ
$\to_{\mathsf{app}}$	x	$(\lambda x.\langle\cdot\rangle x)(\lambda y.y)$	e	$(x, (\lambda x.x\langle\cdot\rangle)(\lambda y.y), e)$
$\to_{\mathsf{var}}$	$\lambda y.y$	$(\lambda x.xx)\langle\cdot\rangle$	ϵ	$(x, (\lambda x.x\langle\cdot\rangle)(\lambda y.y), e)$
$\to_{\mathsf{abs}}$	y	$(\lambda x.xx)(\lambda y.\langle\cdot\rangle)$	$[y{\leftarrow}(x, (\lambda x.x\langle\cdot\rangle)(\lambda y.y), e)]$	ϵ
$\to_{\mathsf{var}}$	x	$(\lambda x.x\langle\cdot\rangle)(\lambda y.y)$	e	ϵ
$\to_{\mathsf{var}}$	$\lambda y.y$	$(\lambda x.xx)\langle\cdot\rangle$	ϵ	ϵ

Optimized KAM.

	Sub-term	Context	Env.	Stack
	$(\lambda x.xx)(\lambda y.y)$	$\langle\cdot\rangle$	ϵ	ϵ
$\to_{\mathsf{app}}$	$\lambda x.xx$	$\langle\cdot\rangle(\lambda y.y)$	ϵ	$(\lambda y.y, (\lambda x.xx)\langle\cdot\rangle, \epsilon)$
$\to_{\mathsf{abs}}$	xx	$(\lambda x.\langle\cdot\rangle)(\lambda y.y)$	$[x{\leftarrow}(\lambda y.y, (\lambda x.xx)\langle\cdot\rangle, \epsilon)]$	ϵ
$\to_{\mathsf{appv}}$	x	$(\lambda x.\langle\cdot\rangle x)(\lambda y.y)$	$[x{\leftarrow}(\lambda y.y, (\lambda x.xx)\langle\cdot\rangle, \epsilon)]$	$(\lambda y.y, (\lambda x.xx)\langle\cdot\rangle, \epsilon)$
$\to_{\mathsf{var}}$	$\lambda y.y$	$(\lambda x.xx)\langle\cdot\rangle$	ϵ	$(\lambda y.y, (\lambda x.xx)\langle\cdot\rangle, \epsilon)$
$\to_{\mathsf{abs}}$	y	$(\lambda x.xx)(\lambda y.\langle\cdot\rangle)$	$[y{\leftarrow}(\lambda y.y, (\lambda x.xx)\langle\cdot\rangle, \epsilon)]$	ϵ
$\to_{\mathsf{var}}$	$\lambda y.y$	$(\lambda x.xx)\langle\cdot\rangle$	ϵ	ϵ